6ELEN018W - Applied Robotics
Lecture 5: Robot Dynamics - Motion upon Forces - Part 1

Dr Dimitris C. Dracopoulos

Kinematics vs Dynamics

- *Kinematic Equations*: describe the motion of a robot without consideration of the forces and torques producing the motion
- *Dynamic Equations*: describe the relationship between force and motion

The equations of motion are important for the:

1. Design of robots
2. Simulation and animation of motion
3. Design of control algorithms for the robot

Newton's Laws of Motion

Newton's 1st Law: Inertia

Every object will remain at rest or in uniform motion in a straight line unless compelled to change its state by the action of an external force.

- This tendency to resist changes in a state of motion is *inertia*
- If all the external forces cancel each other out, then there is no net force acting on the object
- If there is no net force acting on the object, then the object will maintain a constant velocity

Newton's Second Law: Force

$$F = m \cdot \ddot{y}$$

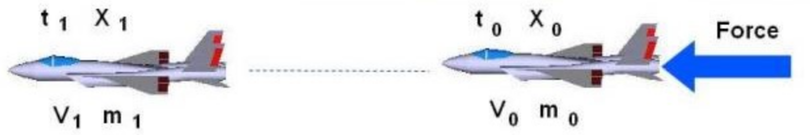
where

- $\ddot{y} = a$ the acceleration of a body
- F is the total force acting on the body

A force is equal to the change of momentum (mass times velocity) per change in time:

$$F = m \frac{V_{t_1} - V_{t_0}}{t_1 - t_0}$$

assuming mass m does not change over time.



Newton's Third Law: Action and Reaction

Whenever one object exerts a force on a second object, the second object exerts an equal and opposite force on the first.

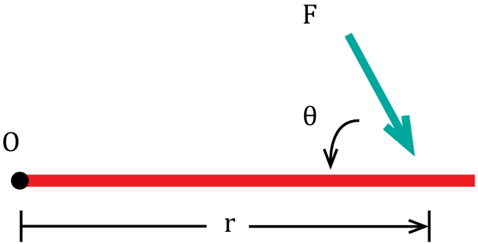
- If object A exerts a force on object B, object B also exerts an equal and opposite force on object A.

Examples:

- The motion of a spinning ball, the air is deflected to one side, and the ball reacts by moving in the opposite direction.
- The motion of a jet engine produces thrust and hot exhaust gases flow out the back of the engine, and a thrusting force is produced in the opposite direction.

Forces and Torques

- Torque (or moment of force) is the rotational analogue of a linear force.



Forces and Torques (cont'd)

The torque describes the rate of change of angular momentum for a body.

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = r \cdot F \cdot \sin\theta \tag{1}$$

where

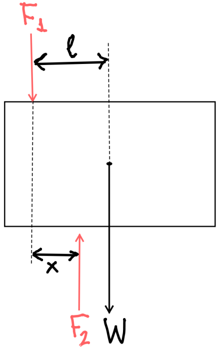
- $\times$ is the cross product (vector product) between two vectors
- $\boldsymbol{r}$ is the position vector from the point that the torque is measured to the point where the force $\boldsymbol{F}$ is applied
- θ is the angle between the force vector and the position vector

Robots Gripping Objects

Currently most industrial robots use 2 fingers to grasp an object.

Example:

A robot tries to hold a rectangular block object with its 2 fingers. The weight force W is applied at the centre of mass. The 2 fingers apply 2 forces F_1 and F_2 at the top and the bottom of the object respectively and these forces are at a distance x.

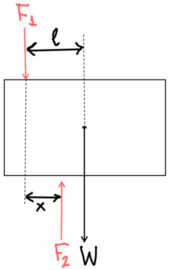


How a Rigid Body Reaches Equilibrium (Balance)?

1. The (vector) sum of all forces should be 0.
2. The sum of the moments of the forces (torques) must equal zero.

Is this sufficient?

The Robot Gripping Example (cont'd)



Apply the equilibrium principles for a rigid body:

What does this mean?

To balance the forces:

$$F_2 = F_1 + W$$

To balance the torques:

$$F_1 * l - F_2 * (l - x) = 0$$

Solve this system of equations for the forces of the 2 robot fingers.

Solution:

$$F_1 + W - F_2 = 0 \Rightarrow F_1 = F_2 - W$$

$$F_1 * l - F_2 * (l - x) = 0 \quad (2)$$

$$(2) \xRightarrow{(1)} (F_2 - W) * l - F_2 * (l - x) = 0 \Rightarrow$$

$$F_2 * l - W * l - F_2 * l + F_2 * x = 0$$

$$\Rightarrow$$

$$\cancel{F_2 * l} - W * l - \cancel{F_2 * l} + F_2 * x = 0$$

$$\Rightarrow$$

$$-W * l + F_2 * x = 0 \Rightarrow$$

$$F_2 = \frac{W * l}{x} \quad (3)$$

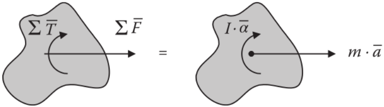
$$\xRightarrow{(1)} F_1 = \frac{W * l}{x} - W \quad (4)$$

Newtonian vs Lagrangian Mechanics

To calculate the dynamic equations of motion for a robot, subject to (generalised) forces, i.e. linear forces and angular forces (torques), one can use either Newtonian or Lagrangian mechanics.

- **Newtonian Mechanics**
  - Easier for simpler systems.
  - More familiar for some people
  - Calculate the sum of all linear forces and sum of all torques:

$$\sum \bar{F} = m\bar{a}, \quad \sum \bar{T} = I\bar{\alpha}$$



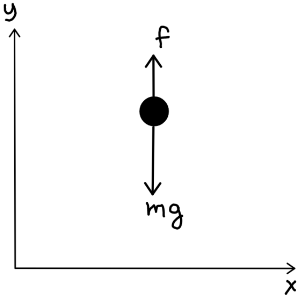
Newtonian vs Lagrangian Mechanics (cont'd)

- **Lagrangian Mechanics**
  - Easier for more complicated systems.
  - Based on system's energies
  - Systematic approach
  - Lagrangian is the difference between kinetic and potential energies of the system:

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

The Euler-Lagrange Equations

Consider a 1-degree of freedom system:



By Newton's second law:

$$m\ddot{y} = f - mg \tag{2}$$

The left hand side can be written as:

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}(\frac{1}{2}m\dot{y}^2) = \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{y}} \tag{3}$$

where $\mathcal{K} = \frac{1}{2}m\dot{y}^2$ is the kinetic energy.

The Euler-Lagrange Equations (cont'd)

The gravitational force in (2) can be written as:

$$mg = \frac{\partial}{\partial y}(mgy) = \frac{\partial \mathcal{P}}{\partial y} \tag{4}$$

where $\mathcal{P} = mgy$ is the potential energy due to gravity.
The difference between the kinetic and the potential energy is called the **Lagrangian** $\mathcal{L}$ of the system:

$$\mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2}m\dot{y}^2 - mgy \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} \overset{(5)}{=} \frac{\partial \mathcal{K}}{\partial \dot{y}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} \overset{(5)}{=} -\frac{\partial \mathcal{P}}{\partial y} \tag{6}$$

Equation (2) can be written as the **Euler-Lagrange equation**:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f \tag{7}$$

How to Use the Euler-Lagrange Equation?

To calculate the dynamic equations of motion for a system such as a serial-link robot, the Euler Lagrange equation can be used:

- Write the kinetic and potential energies of the system in terms of a set of *generalised coordinates* $(q_1, q_2, \ldots q_n)$ where n is the degrees of freedom of the system. q_k can be linear distances or angles:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k, \quad k = 1, \ldots, n$$

  where τ_k is the (generalised) force (linear force or torque) associated with q_k

This is a system of coupled second-order differential equations that can be solved using numerical methods.

How to Use the Euler-Lagrange Equation? (cont'd)

For example, using the Euler-Lagrange equation for a system with both linear forces and angular forces (torques) we can write:

$$F_i = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} \tag{8}$$

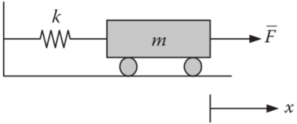
$$T_i = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} \tag{9}$$

where $\mathcal{L}$ is the Lagrangian:

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

An Example of Deriving the Dynamic Equations of Motion

Derive the dynamic equations of motion for the 1-DOF cart-spring shown below, using both Lagrangian mechanics as well as Newtonian mechanics.



The motion of the cart is constrained along the x-axis. Because this is a 1-DOF system there is only one equation describing the linear motion.
Only equation (8) is used and not (9) since there is no angular motion.

An Example of Deriving the Dynamic Equations of Motion

Euler-Lagrange method:

- Kinetic energy:

$$\mathcal{K} = \frac{1}{2} m v^2$$

- Potential energy:

$$\mathcal{P} = \frac{1}{2} k x^2$$

- Lagrangian:

$$\mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2} m v^2 - \frac{1}{2} k x^2$$

- Lagrangian derivatives:

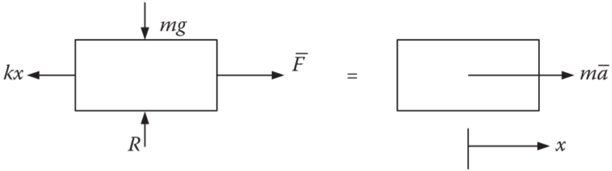
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}(m\dot{x}) = m\ddot{x}, \quad \frac{\partial \mathcal{L}}{\partial x} = -kx$$

- The equation of motion for the cart:

$$F = m\ddot{x} + kx$$

An Example of Deriving the Dynamic Equations of Motion (cont'd)

Newtonian method:



$$\sum \bar{F} = m\bar{a} \Rightarrow F - kx = ma_x \Rightarrow F = ma_x + kx$$

which is the same formula which was derived using the Euler-Lagrange equation.