5ELEN018W - Robotic Principles
Lecture 5: Inverse Kinematics

Dr Dimitris C. Dracopoulos

The problem of Forward kinematics

Kinematic chain: a series of rigid bodies (e.g. links of a robotic arm) connected together by joints.
The joint angles of a kinematic chain determine the position and orientation of the end effector.

- A coordinate frame i relative to coordinate frame $i-1$ is denoted by ${}^{i-1}\boldsymbol{T}_i$:

$$\begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & r_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & r_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

where $\theta_i, \alpha_i, r_i, d_i$ are the DH parameters.

The Problem of Forward Kinematics (cont'd)

The problem of forward kinematics is expressed as the calculation of the transformation between a coordinate frame fixed in the *end-effector* and another coordinate frame fixed in the base.
For example, for 6-joint manipulator:

$$^{0}\boldsymbol{T}_6 = {}^{0}\boldsymbol{T}_1 \cdot {}^{1}\boldsymbol{T}_2 \cdot {}^{2}\boldsymbol{T}_3 \cdot {}^{3}\boldsymbol{T}_4 \cdot {}^{4}\boldsymbol{T}_5 \cdot {}^{5}\boldsymbol{T}_6 \tag{2}$$

The Problem of Inverse Kinematics

Given a position and orientation of a robot's end-effector, calculate the angles θ of the joints.

- Solving the kinematics equations of a manipulator robot is a nonlinear problem.
- Given the homogeneous matrix of the end-effector with respect to the base frame, solve for all the joint angles $\theta_1, \theta_2, \ldots, \theta_n$.

Challenging mathematical problem due to:

1. nature of the nonlinear equations. Often no analytic solutions (closed form) can be calculated and numerical methods are required.
2. often, there are multiple solutions (i.e. multiple sets of joint angles) that can place the end effector at the desired position.

   $\longrightarrow$ The algorithm must choose the solution that results in the most natural and efficient motion of the robot.
3. It is possible that no solutions exist

Applications of Inverse Kinematics

- Manufacturing and assembly
- Surgery
- Search and rescue

Methods for Solving Inverse Kinematics

- Closed-form methods
- Iterative methods

Closed-form solutions are desirable because they are faster than numerical solutions and identify all possible solutions.

- They are not general, but robot dependent.
- To calculate, they take advantage of particular geometric features of specific robot mechanisms.
- As the number of joints increases, this becomes increasingly difficult.
- For some serial-link robot manipulators, no analytical (closed form) solution exists!

Closed-form Methods

- Algebraic methods
- Geometric methods

Algebraic Methods for Solving Inverse Kinematics

1. Identify the significant equations containing the joint variables.
2. Manipulate them into a soluble form.

- A common strategy is reduction to a equation in a single variable, e.g.

$$C_1 \cos \theta_i + C_2 \sin \theta_i + C_3 = 0$$

where C_1, C_2, C_3 are constants.

Solution:

$$\theta_i = 2 \tan^{-1} \left(\frac{C_2 \pm \sqrt{C_2^2 - C_3^2 + C_1^2}}{C_1 - C_3} \right)$$

Algebraic Methods for Solving Inverse Kinematics (cont'd)

- Another useful strategy is the reduction to a pair of equations having the form:

$$\begin{aligned} C_1 \cos\theta_i + C_2 \sin\theta_i + C_3 &= 0 \\ C_1 \sin\theta_i - C_2 \cos\theta_i + C_4 &= 0 \end{aligned}$$

only one solution:

$$\theta_i = \text{atan2}(-C_1 C_4 - C_2 C_3, C_2 C_4 - C_1 C_3)$$

Geometric Methods for Solving Inverse Kinematics

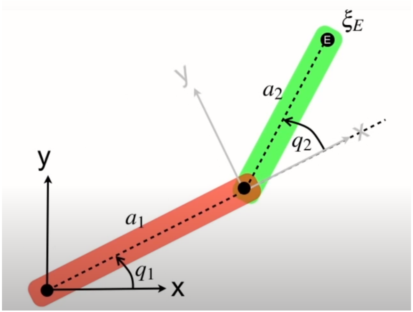
Such methods involve identifying points on the manipulator relative to which position and/or orientation can be expressed as a function of a reduced set of the joint variables using trigonometric relationships.
$\longrightarrow$ often results to the decomposition of the spatial problem into separate planar problems.

- Decomposition of the full problem into inverse position kinematics and inverse orientation kinematics.
- The solution is derived by rewriting equation (2) as:

$$^{0}\boldsymbol{T}_6 \cdot {}^{6}\boldsymbol{T}_5 \cdot {}^{5}\boldsymbol{T}_4 \cdot {}^{4}\boldsymbol{T}_3 = {}^{0}\boldsymbol{T}_1 \cdot {}^{1}\boldsymbol{T}_2 \cdot {}^{2}\boldsymbol{T}_3 \qquad (3)$$

Calculating Analytical Solutions in the Python Robotics Toolbox

Consider a 2-joint Planar (2D) Robot



Given the position of the end-effector (x_E, y_E) calculate the required joint angles to achieve this position.

Calculating Analytical Solutions in Python (con'd)

```
# create symbols for lengths of the 2 links
>>> a1 = Symbol('a1')
>>> a2 = Symbol('a2')

# symbols for joints angles
>>> q1, q2 = symbols("q1:3")

# transformation to calculate the position of the end-effector
>>> e = ET2.R()*ET2.tx(a1)*ET2.R()*ET2.tx(a2)

# calculate forward kinematics matrix for end-effector
>>> TE = e.fkine([q1, q2])

# translation part of matrix gives the position (x_fk, y_fk) of the
# end-effector
>>> x_fk, y_fk  = TE.t
>>> print(x_fk)
>>> print(y_fk)

# create symbolic variables to represent the position of the end-effector
>> x, y = symbols("x, y")

# x_fk = x and y_fk = y
# then x_fk**2 + y_fk**2 - x**2 - y**2 = 0
>>> eq1 = (x_fk**2 + y_fk**2 - x**2 - y**2).trigsimp()
```

Calculating Analytical Solutions in Python (con'd)

```
>>> print(eq1)
a1**2 + 2*a1*a2*cos(q2) + a2**2 - x**2 - y**2

>>> import sympy  # explicitly show that solve belongs to sympy
>>> q2_sol = sympy.solve(eq1, q2)  # 2 solutions exist for q2
>>> print(q2_sol)
[-acos(-(a1**2 + a2**2 - x**2 - y**2)/(2*a1*a2)) + 2*pi,
 acos((-a1**2 - a2**2 + x**2 + y**2)/(2*a1*a2))]


# expand the two equations x_fk=x,  y_fk=y
>>> eq2 = tuple(map(sympy.expand_trig, [x_fk - x,  y_fk - y]))
>>> print(eq2)

(a1*cos(q1) + a2*(-sin(q1)*sin(q2) + cos(q1)*cos(q2)) - x,
a1*sin(q1) + a2*(sin(q1)*cos(q2) + sin(q2)*cos(q1)) - y)

# solve for sin(q1), cos(q1)
>>> q1_sol = sympy.solve(eq2, [sympy.sin(q1), sympy.cos(q1)])
>>> print(q1_sol)  # dictionary containing sin(q1) and cos(q1)
```

Calculating Analytical Solutions in Python (con'd)

```
# tan(q1) = sin(q1)/cos(q1)
>>> print(q1_sol[sin(q1)]/q1_sol[cos(q1)])

# solve for q1
>>> sympy.atan2(q1_sol[sin(q1)], q1_sol[cos(q1)]).simplify()
```

Iterative (Numerical) Methods for Solving Inverse Kinematics

Can be applied to any kinematic robot structure (not robot dependent).

- Slower
- In some cases they do not compute all possible solutions
- Refining the solution through iterations
- Initial starting point affects the solution time

How? Minimise the error between the forward kinematics solution and the desired end-effector pose $\boldsymbol{\xi}_E$:

$$\boldsymbol{q}^* = \arg\min_{\boldsymbol{q}} (FK(\boldsymbol{q}) - \boldsymbol{\xi}_E)$$

Numerical Methods for Solving Inverse Kinematics (cont'd)

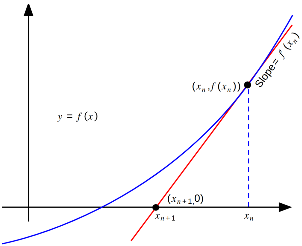
Various classical numerical methods can be applied, including among others:

- Newton-Raphson: first order approximation of original equations
- Levenberg–Marquardt optimisation: using the second order derivative for the approximation of the original system.

The Newton-Raphson Algorithm

The slope (tangent) of a function $f(x)$ for $x = x_n$ is defined (calculated) by the derivative of the function at that point:

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}} \tag{4}$$



Then the approximated solution for finding the root of f (where $f(x) = 0$) can be calculated iteratively by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{5}$$

Calculating Numerical Solutions in Python

```
>>> a1 = 1; a2 = 1
>>> q1, q2 = symbols("q1:3")
>>> e = ET2.R()*ET2.tx(a1)*ET2.R()*ET2.tx(a2)

# Desired position of the end-effector
>>> des_pos = np.array([0.5, 0.4])

# define the error (E) function
>>> def E(q):
>>>     return np.linalg.norm(e.fkine(q).t - des_pos)

# Minimise the error (E) between the forward kinematics solution and the
# desired position of the end-effector - Use optimize from SciPy
>>> sol = optimize.minimize(E, [0, 0])

>>> print(sol.x)   # required q values to achieve des_pos for end-effector
[1.91964289 3.7933814 ]

# Computing the forward kinematics confirms that the
# solution is correct - Recall that we started the
# calculation for des_pos = np.array([0.5, 0.4])
>>> e.fkine(sol.x).printline()

t = 0.5, 0.4; -32.7°
```

Other topics/issues in Robotics

1. **Forward Instantaneous Kinematics**
   - → Given all members of the kinematic chain and the rates of motion about all joints, find the total velocity of the end-effector.
   - → Usage of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$

   $$^k\boldsymbol{v}_N = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{6}$$

   where $^k\boldsymbol{v}_N$ is the velocity of the end-effector expressed in any frame k

2. **Inverse Instantaneous Kinematics**
   - → Given the positions of all the members of the kinematic chain and the total velocity of the end-effector, find the rates of the motion of all joints.
   - → Usage of the inverse of the Jacobian matrix

   $$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v}_n \tag{7}$$