# 5ELEN018W - Robotic Principles
# Lecture 4: Kinematics

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# More On Transformations

- Transformations of a frame (object, or point) which are relative to the fixed reference frame

*Pre-multiply* the transformation matrix with the coordinates described in the moving frame:

$$\boldsymbol{p}_{xyz} = \boldsymbol{Transf}_{xyz} \times \boldsymbol{p}_{x'y'z'} \tag{1}$$

where $xyz$ is the fixed reference frame and $x'y'z'$ is the moving frame.
e.g. for a rotation about the $z$ axis followed by a translation about the $x$ axis, followed by a rotation about the $y$ axis:

$$\boldsymbol{Transform}_{xyz} = \boldsymbol{R}_y \cdot \boldsymbol{T}_x \cdot \boldsymbol{R}_z$$

# More On Transformations (cont'd)

- Transformations of a frame (object, or point) which are relative to the moving reference frame

*Post-multiply* the transformation matrix with the coordinates described in the moving (current) frame:

$$\boldsymbol{p}_{xyz} = \boldsymbol{Transf}_{x'y'z'} \times \boldsymbol{p}_{x'y'z'} \tag{2}$$

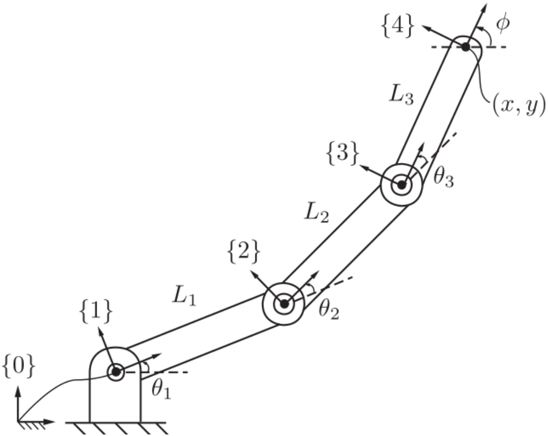
where $xyz$ is the fixed reference frame and $x'y'z'$ is the moving frame.
e.g. for a rotation about the $z'$ axis followed by a translation about the $x'$ axis, followed by a rotation about the $y'$ axis:

$$\boldsymbol{Transform}_{x'y'z'} = \boldsymbol{R}_{z'} \cdot \boldsymbol{T}_{x'} \cdot \boldsymbol{R}_{y'}$$

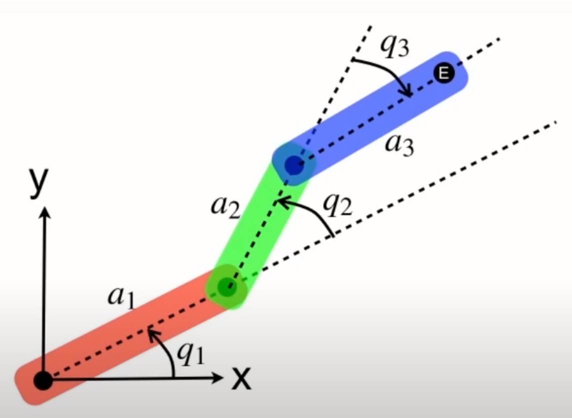
# Forward Kinematics vs Inverse Kinematics

- *Forward Kinematics*: the calculation of the position and orientation of a robot's end-effector from its joint coordinates $\theta$.
- *Inverse Kinematics*: given a position and orientation of a robot's end-effector, calculate the angles $\theta$ of the joints.

# Forward Kinematics



# Forward Kinematics



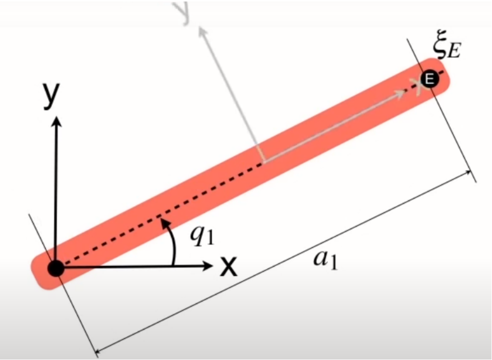
# Representation of Configuration Space of a Robot

The position and orientation of all links.
The pose of the end-effector (i.e. location and orientation) can be described with basic transformation matrices that can be multiplied together to get the homogeneous matrix.

$$HomogeneousMatrix = Transf_1 * Transf2 * Transf3 * \ldots * Transf_n \tag{3}$$

where $n$ is the number of links (assuming that each of these matrices is the <u>total</u> transformation for each link).

# Example of a 1-joint Robot Arm




Rotation by angle $q_1$ and then translation by $a_1$.
The homogeneous transformation describing the overall result can be calculated using the following:

$$EndEffector = Rot(q_1) \cdot T_x(a_1) \tag{4}$$

# Pose of the End-Effector - 1-Joint 2D Robot Arm (cont'd)

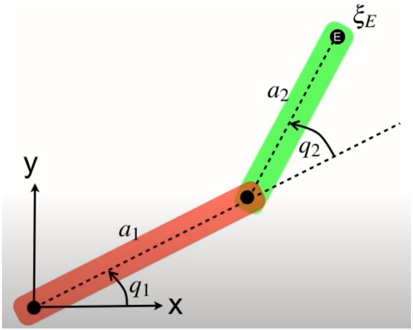
In Python Robotics Toolbox:

```
>>> from sympy import *

>>> q1 = Symbol('q1')

>>> trot2(q1)

>>> a1=Symbol('a1')

>>> transl2(a1,0)

>>> E = trot2(q1) @ transl2(a1, 0)
```

or equivalently as a ETS2 object:

```
>>> e = ET2.R()*ET2.tx(a1)

>> e.plot(0)  # plot the ETS2 object with q1 = 0 degrees
>> e.plot(math.pi/4) # plot the ETS2 object with q1 = 45 degrees
```

# Example of a 2-joint Planar Robot Arm




1. Rotation by angle $q_1$
2. Translation by $a_1$
3. Rotation by angle $q_2$
4. Translation by $a_2$

The homogeneous transformation describing the overall result can be calculated using the following:

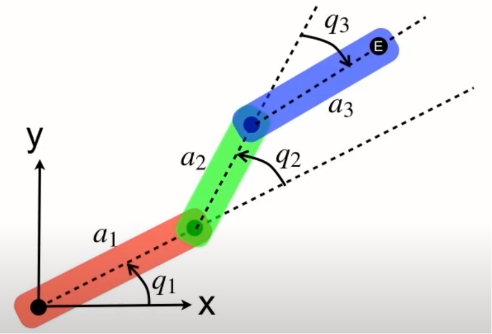
$$EndEffector = Rot(q_1) \cdot T_x(a_1) \cdot Rot(q_2) \cdot T_x(a_2) \tag{5}$$

# Pose of the End-Effector - 2-Joint 2D Robot Arm (cont'd)

In Python Robotics Toolbox:

```
>>> from sympy import *

>>> q1 = Symbol('q1')

>>> trot2(q1)

>>> a1=Symbol('a1')

>>> transl2(a1,0)

>>> q2 = Symbol('q2')

>>> a2 = Symbol('a2')

>>> E = trot2(q1) @ transl2(a1, 0) @ trot2(q2) @ transl2(a2, 0)

E = simplify(E)
```

# Example of a 3-joint Planar Robot Arm




The homogeneous transformation describing the overall result can be calculated using the following:

$$EndEffector = Rot(q_1) \cdot T_x(a_1) \cdot Rot(q_2) \cdot T_x(a_2) \cdot Rot(q_3) \cdot T_x(a_3)$$

# Pose of the End-Effector - 3-Joint 2D Robot Arm (cont'd)

In Python Robotics Toolbox:

```
>>> from sympy import *

>>> q1 = Symbol('q1')
>>> trot2(q1)

>>> a1=Symbol('a1')
>>> transl2(a1,0)

>>> q2 = Symbol('q2')
>>> a2 = Symbol('a2')

>>> q3 = Symbol('q3')
>>> a3 = Symbol('a3')

>>> E = trot2(q1)@transl2(a1, 0)@trot2(q2)@transl2(a2, 0) \
                  @ trot2(q3) @ transl2(a3, 0)
E = simplify(E)
```

# The Problem of Forward Kinematics

The calculation of the position and orientation of a robot's end-effector from its joint coordinates $\theta_i$.

- In the previous slides it has been shown how to do this in 2D spaces for:
  - $\rightarrow$ 1-joint robot arms
  - $\rightarrow$ 2-joint robot arms
  - $\rightarrow$ 3-joint robot arms

using simple transformations in Mathematics which correspond to real operations in Physics!

# The Denavit-Hartenberg (DH) Notation

The relationship between two coordinate frames is described by 6 parameters (3 translations and 3 rotations). Can this be improved?

Attach a coordinate frame to the end of each link.

- Reduces the relationship between 2 coordinate frames from 6 parameters to 4 parameters.
- Each joint in a robot is described by 4 parameters.

*How is this achieved?*
The coordinate frames have constraints.

- $x$ axis of frame $j$ intersects the $z$ axis of frame $j-1$
- $x$ axis of frame $j$ is perpendicular to the $z$ axis of frame $j-1$

$\longrightarrow$ 6 parameters - 2 constraints means 4 parameters are needed.

# The Denavit-Hartenberg (DH) Notation (cont'd)

4 parameters used associated with each link $i$ and joint $i$:

- $\boldsymbol{\theta_i}$: joint angle
- $\boldsymbol{d_i}$: link offset
- $\boldsymbol{r_i}$ (or $\boldsymbol{a_i}$ in most textbooks): link length
- $\boldsymbol{\alpha_i}$: link twist

Each homogeneous transformation $A_i$ is represented as the product of 4 basic transformations:

$$A_i = Rot_{z,\theta_i} \cdot Trans_{z,d_i} \cdot Trans_{x,r_i} \cdot Rot_{x,\alpha_i} \tag{6}$$

# The Denavit-Hartenberg (DH) Notation (cont'd)

$$A_i = Rot_{z,\theta_i} \cdot Trans_{z,d_i} \cdot Trans_{x,r_i} \cdot Rot_{x,\alpha_i} =$$

$$\begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) & 0 \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & r_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & r_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

# The DH Table

The DH notation requires a table. The number of rows equals the number of joints and it has 4 columns each one corresponding to the 4 parameters for the joint $i$ of that row.

For example for a robot with 4 joints:

| **Joint** | $\boldsymbol{\theta}$ | $\boldsymbol{r}$ | $\boldsymbol{d}$ | $\boldsymbol{\alpha}$ |
|---|---|---|---|---|
| 1 | $\theta_1$ | $r_1$ | $d_1$ | $\alpha_1$ |
| 2 | $\theta_2$ | $r_2$ | $d_2$ | $\alpha_2$ |
| 3 | $\theta_3$ | $r_3$ | $d_3$ | $\alpha_3$ |
| 4 | $\theta_4$ | $r_4$ | $d_4$ | $\alpha_4$ |

- For *revolute joints*: Only $\theta$ changes, all the other 3 parameters are fixed according to the robot mechanism.
- For *prismatic joints*: Only $d$ changes, all the other 3 parameters are fixed according to the robot mechanism.

# Example of DH Notation

Consider the following DH table:

| **Joint** | $\boldsymbol{\theta}$ | $\boldsymbol{r}$ | $\boldsymbol{d}$ | $\boldsymbol{\alpha}$ |
|---|---|---|---|---|
| 1 | $\pi$ | 5 | 2 | $\frac{\pi}{2}$ |

What is the DH matrix which corresponds to the above table?

*Answer:*

```
   -1.0000   -0.0000   -0.0000   -5.0000
    0.0000   -0.0000    1.0000    0.0000
         0    1.0000    0.0000    2.0000
         0         0         0    1.0000
```

To calculate, apply Equation (7).

# Finding the Pose of the End-Effector relative to the Base Frame

Assume that $A_1, A_2, A_3 \ldots A_n$ are the DH matrices of all the robot joints $1, 2, 3, \ldots n$.
Then the calculation requires the multiplication of all the matrices:

$$Pose_{end\_effector} = A_1 \cdot A_2 \cdot A_3 \ldots A_n \tag{8}$$

# Example: Calculation of the Pose of the End-Effector

The following DH matrices correspond to the joints of a robot, from robot base to end-effector. Find the pose of the end-effector relative to the robot base.

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -0 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Example: Calculation of the Pose of the End-Effector (cont'd)

Simply calculate $A_1 * A_2 * A_3$.