5ELEN018W - Robotic Principles
Lecture 4: Kinematics

Dr Dimitris C. Dracopoulos

More On Transformations

- Transformations of a frame (object, or point) which are relative to the fixed reference frame

Pre-multiply the transformation matrix with the coordinates described in the moving frame:

$$\boldsymbol{p}_{xyz} = \boldsymbol{Transf}_{xyz} \times \boldsymbol{p}_{x'y'z'} \tag{1}$$

where xyz is the fixed reference frame and $x'y'z'$ is the moving frame.
e.g. for a rotation about the z axis followed by a translation about the x axis, followed by a rotation about the y axis:

$$\boldsymbol{Transform}_{xyz} = \boldsymbol{R}_y \cdot \boldsymbol{T}_x \cdot \boldsymbol{R}_z$$

More On Transformations (cont'd)

- Transformations of a frame (object, or point) which are relative to the moving reference frame

Post-multiply the transformation matrix with the coordinates described in the moving (current) frame:

$$\boldsymbol{p}_{xyz} = \boldsymbol{Transf}_{x'y'z'} \times \boldsymbol{p}_{x'y'z'} \tag{2}$$

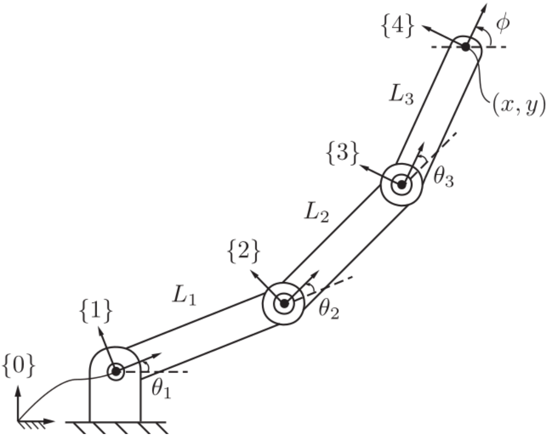
where xyz is the fixed reference frame and $x'y'z'$ is the moving frame.
e.g. for a rotation about the z' axis followed by a translation about the x' axis, followed by a rotation about the y' axis:

$$\boldsymbol{Transform}_{x'y'z'} = \boldsymbol{R}_{z'} \cdot \boldsymbol{T}_{x'} \cdot \boldsymbol{R}_{y'}$$

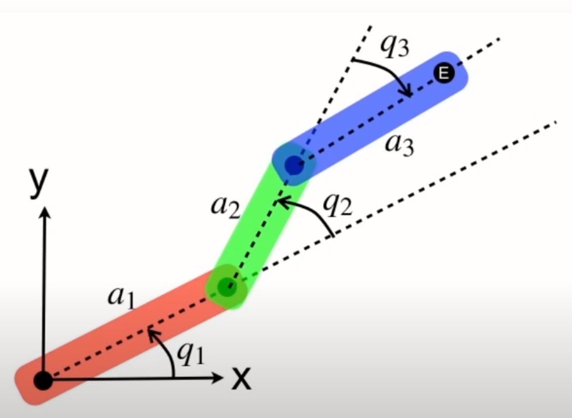
Forward Kinematics vs Inverse Kinematics

- *Forward Kinematics*: the calculation of the position and orientation of a robot's end-effector from its joint coordinates θ.
- *Inverse Kinematics*: given a position and orientation of a robot's end-effector, calculate the angles θ of the joints.

Forward Kinematics



Forward Kinematics



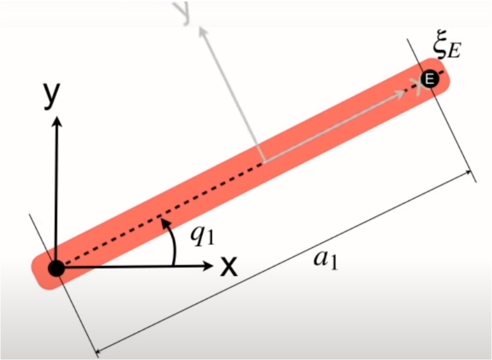
Representation of Configuration Space of a Robot

The position and orientation of all links.
The pose of the end-effector (i.e. location and orientation) can be described with basic transformation matrices that can be multiplied together to get the homogeneous matrix.

$$HomogeneousMatrix = Transf_1 * Transf2 * Transf3 * \ldots * Transf_n \quad (3)$$

where n is the number of links (assuming that each of these matrices is the total transformation for each link).

Example of a 1-joint Robot Arm



Rotation by angle q_1 and then translation by a_1.
The homogeneous transformation describing the overall result can be calculated using the following:

$$EndEffector = Rot(q_1) \cdot T_x(a_1) \tag{4}$$

Pose of the End-Effector - 1-Joint 2D Robot Arm (cont'd)

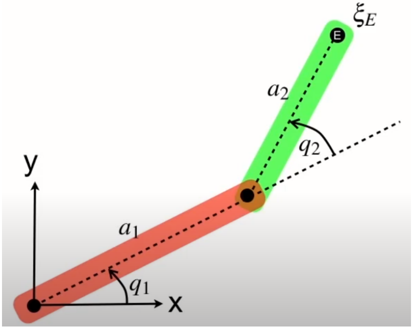
In Python Robotics Toolbox:

```
>>> from sympy import *

>>> q1 = Symbol('q1')

>>> trot2(q1)

>>> a1=Symbol('a1')

>>> transl2(a1,0)

>>> E = trot2(q1) @ transl2(a1, 0)
```

or equivalently as a ETS2 object:

```
>>> e = ET2.R()*ET2.tx(a1)

>> e.plot(0)  # plot the ETS2 object with q1 = 0 degrees
>> e.plot(math.pi/4) # plot the ETS2 object with q1 = 45 degrees
```

Example of a 2-joint Planar Robot Arm



1. Rotation by angle q_1
2. Translation by a_1
3. Rotation by angle q_2
4. Translation by a_2

The homogeneous transformation describing the overall result can be calculated using the following:

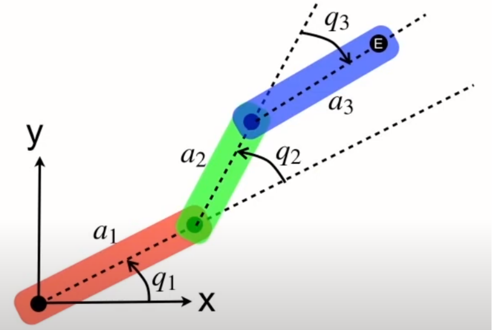
$$EndEffector = Rot(q_1) \cdot T_x(a_1) \cdot Rot(q_2) \cdot T_x(a_2) \qquad (5)$$

Pose of the End-Effector - 2-Joint 2D Robot Arm (cont'd)

In Python Robotics Toolbox:

```
>>> from sympy import *

>>> q1 = Symbol('q1')

>>> trot2(q1)

>>> a1=Symbol('a1')

>>> transl2(a1,0)

>>> q2 = Symbol('q2')

>>> a2 = Symbol('a2')

>>> E = trot2(q1) @ transl2(a1, 0) @ trot2(q2) @ transl2(a2, 0)

E = simplify(E)
```

Example of a 3-joint Planar Robot Arm



The homogeneous transformation describing the overall result can be calculated using the following:

$$EndEffector = Rot(q_1) \cdot T_x(a_1) \cdot Rot(q_2) \cdot T_x(a_2) \cdot Rot(q_3) \cdot T_x(a_3)$$

Pose of the End-Effector - 3-Joint 2D Robot Arm (cont'd)

In Python Robotics Toolbox:

```
>>> from sympy import *

>>> q1 = Symbol('q1')
>>> trot2(q1)

>>> a1=Symbol('a1')
>>> transl2(a1,0)

>>> q2 = Symbol('q2')
>>> a2 = Symbol('a2')

>>> q3 = Symbol('q3')
>>> a3 = Symbol('a3')

>>> E = trot2(q1)@transl2(a1, 0)@trot2(q2)@transl2(a2, 0) \
                  @ trot2(q3) @ transl2(a3, 0)
E = simplify(E)
```

The Problem of Forward Kinematics

The calculation of the position and orientation of a robot's end-effector from its joint coordinates θ_i.

- In the previous slides it has been shown how to do this in 2D spaces for:
  - $\rightarrow$ 1-joint robot arms
  - $\rightarrow$ 2-joint robot arms
  - $\rightarrow$ 3-joint robot arms

using simple transformations in Mathematics which correspond to real operations in Physics!

The Denavit-Hartenberg (DH) Notation

The relationship between two coordinate frames is described by 6 parameters (3 translations and 3 rotations). Can this be improved?

Attach a coordinate frame to the end of each link.

- Reduces the relationship between 2 coordinate frames from 6 parameters to 4 parameters.
- Each joint in a robot is described by 4 parameters.

How is this achieved?

The coordinate frames have constraints.

- x axis of frame j intersects the z axis of frame $j-1$
- x axis of frame j is perpendicular to the z axis of frame $j-1$

$\longrightarrow$ 6 parameters - 2 constraints means 4 parameters are needed.

The Denavit-Hartenberg (DH) Notation (cont'd)

4 parameters used associated with each link i and joint i:

- $\boldsymbol{\theta_i}$: joint angle
- $\boldsymbol{d_i}$: link offset
- $\boldsymbol{r_i}$ (or $\boldsymbol{a_i}$ in most textbooks): link length
- $\boldsymbol{\alpha_i}$: link twist

Each homogeneous transformation A_i is represented as the product of 4 basic transformations:

$$A_i = Rot_{z,\theta_i} \cdot Trans_{z,d_i} \cdot Trans_{x,r_i} \cdot Rot_{x,\alpha_i} \tag{6}$$

The Denavit-Hartenberg (DH) Notation (cont'd)

$$A_i = Rot_{z,\theta_i} \cdot Trans_{z,d_i} \cdot Trans_{x,r_i} \cdot Rot_{x,\alpha_i} =$$

$$\begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) & 0 \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & r_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & r_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

The DH Table

The DH notation requires a table. The number of rows equals the number of joints and it has 4 columns each one corresponding to the 4 parameters for the joint i of that row.

For example for a robot with 4 joints:

Joint	$\boldsymbol{\theta}$	$\boldsymbol{r}$	$\boldsymbol{d}$	$\boldsymbol{\alpha}$
1	θ_1	r_1	d_1	α_1
2	θ_2	r_2	d_2	α_2
3	θ_3	r_3	d_3	α_3
4	θ_4	r_4	d_4	α_4

- For *revolute joints*: Only θ changes, all the other 3 parameters are fixed according to the robot mechanism.
- For *prismatic joints*: Only d changes, all the other 3 parameters are fixed according to the robot mechanism.

Example of DH Notation

Consider the following DH table:

Joint	$\boldsymbol{\theta}$	$\boldsymbol{r}$	$\boldsymbol{d}$	$\boldsymbol{\alpha}$
1	π	5	2	$\frac{\pi}{2}$

What is the DH matrix which corresponds to the above table?

Answer:

```
-1.0000   -0.0000   -0.0000   -5.0000
 0.0000   -0.0000    1.0000    0.0000
      0    1.0000    0.0000    2.0000
      0         0         0    1.0000
```

To calculate, apply Equation (7).

Finding the Pose of the End-Effector relative to the Base Frame

Assume that $A_1, A_2, A_3 \dots A_n$ are the DH matrices of all the robot joints $1, 2, 3, \dots n$.
Then the calculation requires the multiplication of all the matrices:

$$Pose_{end\_effector} = A_1 \cdot A_2 \cdot A_3 \dots A_n \tag{8}$$

Example: Calculation of the Pose of the End-Effector

The following DH matrices correspond to the joints of a robot, from robot base to end-effector. Find the pose of the end-effector relative to the robot base.

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -0 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Calculation of the Pose of the End-Effector (cont'd)

Simply calculate $A_1 * A_2 * A_3$.