5ELEN018W - Robotic Principles Lecture 3: Position and Orientation: Transformations

Dr Dimitris C. Dracopoulos

Pose

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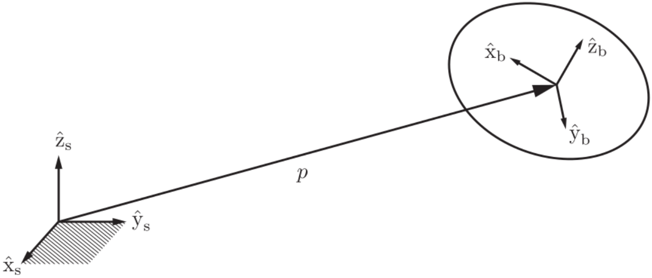
- Multiple coordinate frames are used in robotics to facilitate the computations for motion and different types of functionality.
- NASA is using them to simplify calculations!



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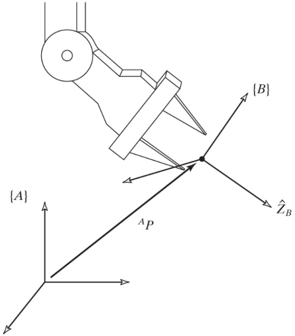
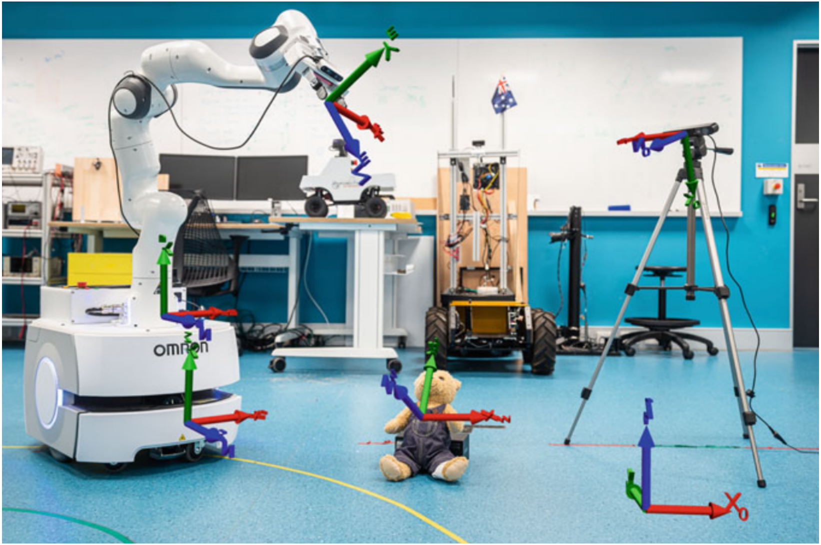


Reference Frames in Real World Robots



How to Specify Pose

Using transformations:

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Using transformations:

- Rotation
  - → represents orientation
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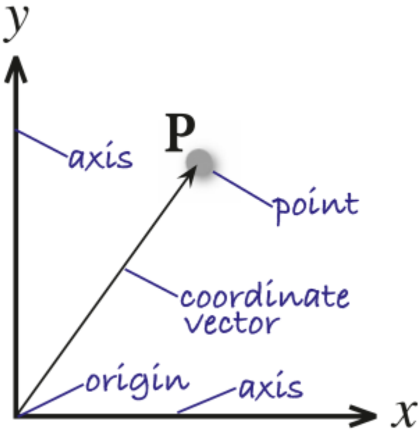
  - → rotate a vector or a frame
- Translation (linear move along one of the axes)

Terminology of Coordinate Frames



Pose in the 2D Space

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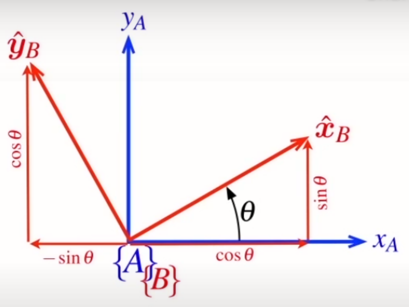

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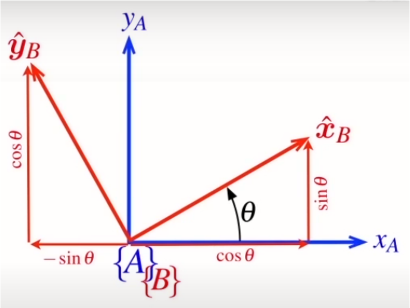

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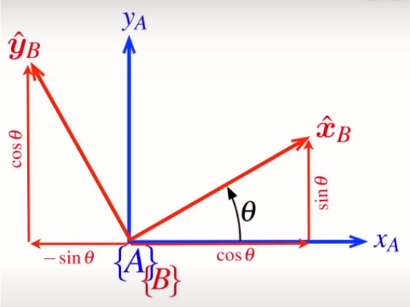

- A new coordinate frame $\{B\}$ with the same origin as $\{A\}$ but rotated counter-clockwise by angle θ (positive angle)
- Transforms vectors (their coordinates) from new frame $\{B\}$ to the old frame $\{A\}$:

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} \quad (1)$$

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- The determinant is 1: $det(\boldsymbol{R}) = 1$
  $\rightarrow$ the length of a vector is unchanged after the rotation

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How to Install the Additional (non-Mathworks) Matlab Robotics Toolbox

Download from
`https://github.com/petercorke/RVC3-MATLAB` the zip file and unzip it in a location where you can use it (in the university labs this should be done in your `H:` drive).
The `toolbox` directory in the directory you unzipped the above should be included in the Matlab path:

- In the university labs and from inside Matlab execute `path(path,'H:\RVC3-MATLAB\toolbox')` assuming that `H:\RVC3-MATLAB` contains the extracted contents of the zip file. You have to execute this every time you restart Matlab in the labs.
- In your personal computer where you have admin rights, you can use the `pathtool` command (from inside Matlab) to add the additional directory to the Matlab path permanently.

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Assuming P is the position of some object in a 2D space then we can apply transformation $\boldsymbol{T_V}$ by simply adding V to P:

$$\boldsymbol{T_V}(\boldsymbol{P}) = \boldsymbol{P} + \boldsymbol{V} \tag{3}$$

Homogeneous Form

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A row $[0, 0, 1]$ is appended in the end.

Derivation of the Homogeneous Form

$$
\begin{aligned}
\begin{pmatrix} A_x \\ A_y \end{pmatrix} &= \begin{pmatrix} A'_x \\ A'_y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \\
&= \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} {}^B x \\ {}^B y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \\
&= \begin{pmatrix} cos\theta & -sin\theta & t_x \\ sin\theta & cos\theta & t_x \end{pmatrix} \begin{pmatrix} {}^B x \\ {}^B y \\ 1 \end{pmatrix}
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$$= \begin{pmatrix} cos\theta & -sin\theta & t_x \\ sin\theta & cos\theta & t_x \end{pmatrix} \begin{pmatrix} {}^Bx \\ {}^By \\ 1 \end{pmatrix}$$

or equivalently:

$$\begin{pmatrix} A_x \\ A_y \\ 1 \end{pmatrix} = \begin{pmatrix} {}^A\boldsymbol{R}_B(\theta) & {}^A\boldsymbol{t}_B \\ \mathbf{0}_{1\times 2} & 1 \end{pmatrix} \begin{pmatrix} {}^Bx \\ {}^By \\ 1 \end{pmatrix} \qquad (4)$$

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- The homogeneous transformation can be considered as the relative pose which first translates the coordinate frame by ${}^{A}\boldsymbol{t}_B$ with respect to frame $\{A\}$ and then is rotated by ${}^{A}\boldsymbol{R}_B(\theta)$

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Translation of (1, 2) followed by a rotation of $\frac{\pi}{4}$ radians:

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or in a single step:

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A coordinate frame representing the above pose can be plotted:

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>> axis([0 5 0 5]) range of values in both axes is [0, 5]
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>> plottform2d(TAB,frame="AB",color="g");
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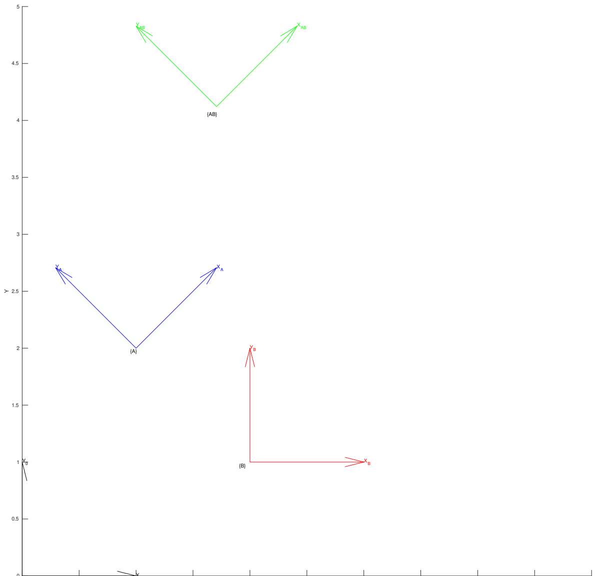
Working with Matlab (cont'd)



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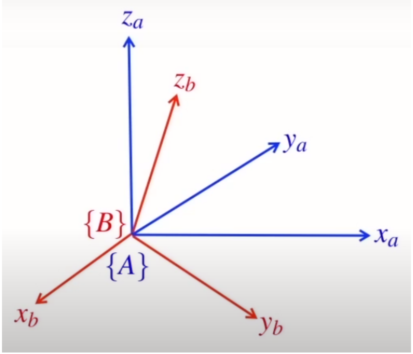

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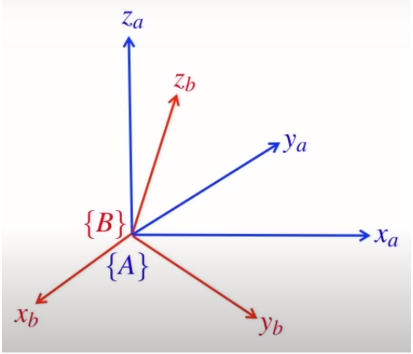

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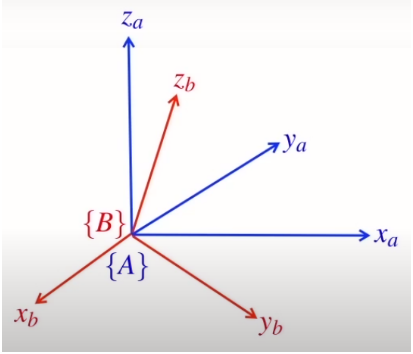

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Rotation about the x-axis:

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Rotation about the x-axis:

$$\boldsymbol{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix} \tag{5}$$

Elementary Rotation Matrices in 3D

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Rotation about the y-axis:

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Rotation about the y-axis:

$$\boldsymbol{R}_y(\theta) = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix} \tag{6}$$

Elementary Rotation Matrices in 3D

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Rotation about the y-axis:

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Elementary Rotation Matrices in 3D

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- Rotations in 3D are not commutative (the order of rotation matters!)

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- Matlab is using a row representation for this.

Matlab Example

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>> a=[1    0          0
      0    cos(pi)    -sin(pi)
      0    sin(pi)    cos(pi)    ]

a =

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>> rotm2axang(a)

ans =

    1.0000         0         0    3.1416
```

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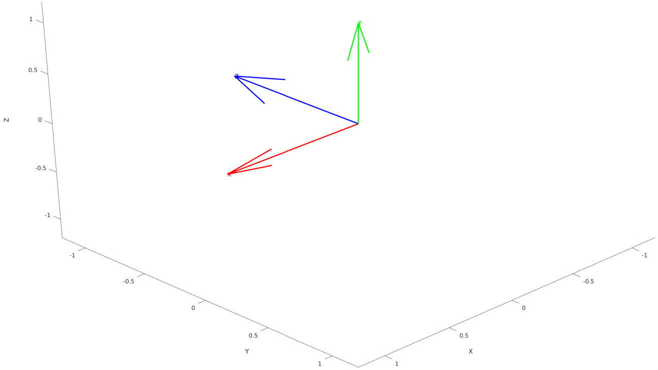
To animate the reference frame moving to the specified relative pose:

```
>> animtform(R)
```


Z
1
0.5
0
-0.5
-1
Y
-1
-0.5
0
0.5
1
X
-1
-0.5
0
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1

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Assuming P is the position of some object then we can apply transformation $\boldsymbol{T_V}$ by simply adding V to P:

$$\boldsymbol{T_V}(\boldsymbol{P}) = \boldsymbol{P} + \boldsymbol{V} \tag{10}$$

Representing Pose in 3D

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Different ways:

- Vector and 3 angles (roll, pitch, yaw)
- Homogeneous transformation (rotation and translation)
  - $\rightarrow$ advantage of transformations calculations using matrix multiplications!

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Construct a 4×4 array with the rotation matrix with 3 zeros (0) in the row below it, and the translation vector with an extra element of 1, as a column next to the rotation matrix:

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$\longrightarrow$ Remember, the matrix-based transformations allow to apply them (<u>or even to combine them!</u>) using **matrix multiplication**!

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Although the inverse of the homogeneous transformation can be calculated as normally by computing the inverse of the original matrix (transformation), this can be done much faster.

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- The homogeneous transformation matrix can be written as:

$$\begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

  where R is the rotation matrix part and d is the translation vector part.

- then the inverse of the matrix (transformation) can be calculated as:

$$\begin{bmatrix} R' & -R' * d \\ 0 & 1 \end{bmatrix}$$